Math 600 Day 8: Vector Fields

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Outline

1. Vector Fields
   - Vector fields on smooth manifolds.
   - Integral curves of vector fields.

Vector Fields

Vector fields in Euclidean space $\mathbb{R}^n$.

Pick and fix a basis $e_1, ..., e_n$ for $\mathbb{R}^n$. Then we get a corresponding basis $(e_1)_p, ..., (e_n)_p$ for each tangent space $\mathbb{R}^n_p$.

A **vector field** V on $\mathbb{R}^n$ is a selection of a tangent vector $V(p) \epsilon \mathbb{R}^n_p$ for each point $p \epsilon \mathbb{R}^n$. We can write

$$V(p) = v^1(p)(e_1)_p + ... + v^n(p)(e_n)_p.$$

The real-valued functions $v^i : \mathbb{R}^n \to \mathbb{R}$ are called the component functions of V.

Example. $V = -y\mathbf{i} + x\mathbf{j}$ on $\mathbb{R}^2$.

Alternative Definition

A **vector field** is a continuous map $V : M \to TM$ with the property that $\pi \circ V = Id_M$.

The vector field is said to be **continuous, differentiable,** etc. if its component functions v_i are. We will usually deal with C^∞ vector fields, so that we can differentiate the component functions as much as we want, and use the word smooth as a synonym for C^∞.

Vector fields can also be defined on open subsets of $\mathbb{R}^n$ in the same way.

Vector fields can be added by adding their values at each point, and multiplied by functions likewise.

Notation.

When working with vector fields on $\mathbb{R}^n$, we sometimes denote the standard basis of $\mathbb{R}^n$ by

$$\frac{\partial}{\partial x_1} = (1, 0, ..., 0), ..., \frac{\partial}{\partial x_n} = (0, ..., 0, 1),$$

and use this same notation at each point $p \epsilon \mathbb{R}^n$.

Then a vector field V on $\mathbb{R}^n$ will be written as

$$V(p) = v^1(p)\frac{\partial}{\partial x_1} + ... + v^n(p)\frac{\partial}{\partial x_n}.$$

Using the Einstein summation convention, we can simply write

$$V(p) = v^i(p)\frac{\partial}{\partial x_i},$$

with the summation over like indices, one upper and one lower, understood.

The above notation suggests that vector fields can serve as differential operators, and we will soon see how this happens, and study it.

Vector fields on smooth manifolds.

A **vector field** V on a smooth manifold M is a choice of a tangent vector $V(p)$ in each tangent space T_pM.

If $f : M \rightarrow N$ is a diffeomorphism, then the vector field V on M gives rise to a vector field $f * V$ on N by defining

$$(f * V)(f(p)) = (df_p)V(p).$$

We call $f * V$ the push forward of V.

Caution. Vector fields push forward under diffeomorphisms, but not in general under smooth maps.

If (U, ϕ) is a coordinate chart on M, then the vector field V restricted to U pushes forward under ϕ to a vector field $f * (V|_U)$ on $\phi(U)$.

Let $\{(U, \phi)\}$ be an atlas for M.

We say that the vector field V on M is smooth if, for each coordinate system (U, ϕ) in this atlas, its push forward to a vector field on $\phi(U)$ is smooth there in the Euclidean sense.

Example.

1. Find two "really different" smooth vector fields on the two-sphere S^2 which vanish (i.e., are zero) at just two points.
2. Find a smooth vector field on S^2 which vanishes at just one point.
3. It is impossible to find a smooth (or even just continuous) vector field on S^2 which never vanishes. The phrase "you can't comb the hair on a billiard ball" refers to this fact. Try to get an intuitive feeling for its validity.
4. Find a smooth vector field on the 3-sphere S^3 which never vanishes.

Remark

A manifold is said to be closed if it is compact and has no boundary.

The only closed 2-manifolds which can support a nowhere-vanishing (continuous or smooth) vector field are the torus and the Klein bottle.

By contrast, every closed 3-manifold can support a nowhere-vanishing (continuous or smooth) vector field.

Definition

If V is a vector field on a smooth manifold M^m and $V(x_0) = 0$, define the **index** of x_0 to be the degree of the map $S_\epsilon \to S^{m-1}$ via $x \to \frac{V(x)}{|V(x)|}$.

Theorem

If V is a smooth vector field on the compact orientable manifold M with only finitely many zeros, then the global sum of the indices of the zeros equals the Euler characteristic of M.

Integral curves of vector fields.

Let M be a smooth manifold, and $\phi : I \to M$ a smooth map of an open interval $I \subset \mathbb{R}$ into M . Then for each $t \epsilon I$ we have the associated linear map of tangent spaces,

$$d\phi_t = (\phi_*)_t : T_t I \to T_{\phi(t)} M.$$

For simplicity, we write

$$(\phi_*)_t(\frac{\partial}{\partial t}) = \frac{d\phi}{dt}$$

and speak of it as the tangent vector (or velocity vector) to our curve at the point $\phi(t) \epsilon M$.

Now suppose that, in addition to a smooth curve $\phi : I \longrightarrow M$ we have a smooth vector field V on M . Suppose, in addition, that for each $t \epsilon I$, we have

$$\frac{d\phi}{dt} = V(\phi(t)).$$

In other words, the tangent vector to our curve ϕ at the point $\phi(t)$ in M coincides with the value of the vector field $V(\phi(t))$ there. Then we say that $\phi : I \longrightarrow M$ is an **integral curve** for the vector field V.

Examples Consider the vector field

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

in the plane $\mathbb{R}^2$, and consider the curves

$$\phi(t) = (a \times cos(t), a \times sin(t))$$

for each real value of $a \geq 0$.

These are all integral curves for the vector field V, and that they fill out the plane.

Note that when $a = 0$, the integral curve is “stationary”, that is, it consists of a single point. This is understandable, since the vector field V is zero at the origin.

We now ask if the simple picture obtained in the preceding example, in which the plane is nicely filled by integral curves of a given vector field, is typical.

Three Questions. Suppose V is a vector field on the smooth manifold M.

1. Is there an integral curve of V through each point of M?
2. If so, can there be more than one integral curve through a point of M?
3. If there is a unique integral curve through each point of M, does that curve vary continuously as the point changes?

Example Interpret the differential equation $\frac{dy}{dx} = -y^2$ as a vector field on the plane, and show that it has integral curves which are not defined for all time.

Example (a) Interpret the differential equation $\frac{dy}{dx} = y^{\frac{2}{3}}$ as a vector field on the plane, and show that it has two different solutions satisfying the initial condition $y(0) = 0$.

Basic existence and uniqueness theorems for ordinary differential equations.

Let U be an open set in $\mathbb{R}^n$, $x_0 \epsilon U$, and $V : U \to \mathbb{R}^n$ a continuous function. We seek a differentiable map $\phi : I \to U$, where I is some open interval in R about 0, satisfying the differential equation

$$\frac{d\phi}{dt} = V(\phi(t)),$$

with initial condition

$$\phi(0) = x_0.$$

Theorem

(a) If V is continuous, then such a map ϕ always exists.

(b) If V is Lipschitz in a neighborhood of x_0, then such a map ϕ is unique, in the sense that any two such maps agree on a neighborhood of 0 in $\mathbb{R}$.

Comment. Traditionally, the above differential equation is written as $\frac{dx}{dt} = V(x)$.

Theorem

Let V be a C^∞ vector field on the C^∞ differentiable manifold M. Then there exists an open neighborhood U of $0 \times M$ in $\mathbb{R} \times M$ and a C^∞ map $\Phi : U \longrightarrow M$ (we will write $\phi_t(x) = \Phi(t, x)$) such that

1. $\phi_0(x) = x$.
2. *$V(x)$ is the tangent vector to the curve $t \rightarrow \phi_t(x)$ at $t = 0$.*
3. *$\phi_s(\phi_t(x)) = \phi_{s+t}(x)$ whenever both sides are defined.*

Addendum. If the vector field V has compact support (in particular, if M is compact), then $U = \mathbb{R} \times M$ and each $\phi_t : M \to M$ is a diffeomorphism.

In that case, the collection $\{\phi_t : t \epsilon \mathbb{R}\}$ is called a **one-parameter group of diffeomorphisms of** M, equivalent, a flow on M, and the vector field V is said to be its **infinitesimal generator**.

Example. Let $M = \mathbb{R}$ and $V = \frac{\partial}{\partial t}$. Then we can take $U = \mathbb{R} \times M$ and the flow is given by $\phi_t(x) = x + t$.